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LETTER TO THE EDITOR

Topology and renormalisability

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Abstract. Using functional methods and the automorphic formalism, it is shown that a field theory on a multiply connected background is renormalisable provided the corresponding theory on the universal covering space is renormalisable.

It is widely recognised that in a full quantum theory of gravity, fluctuations in the topology of space–time, at least at the Planck length scale of distances, are likely to occur. On the other hand, should such a fluctuation happen to have got 'frozen in' at an early stage of the universe's lifetime, the universe would now be topologically non-trivial in the large. Considerations such as these have led in recent years to the exploration of properties of field theories which are a direct consequence of non-trivial space–time topology (see, e.g., the references).

For scalar and spinor fields (the cases most frequently considered) one needs to demand that the space–time is multiply connected and in that case the automorphic formalism (Banach and Dowker 1979a, b, Banach 1980) is available to us in which the situation of interest is pulled back to the universal covering space of the multiply connected space–time. For interacting field problems, renormalisability is certainly an important topic and since renormalisability for the universal covering space is something that may, in specific instances, be decided already, one would hope that the automorphic formalism would be capable of deciding the issue for the factor space too. It is the purpose of this Letter to show that this is indeed the case.

Suppose $T\otimes M$ is our multiply connected space–time (T is time, M is the multiply connected spatial section (this split is (topologically) demanded by the requirement of global hyperbolicity—we may for convenience assume that the metric respects it too)) and let $T\otimes\tilde{M}$ be the universal cover. We recall (Banach and Dowker 1979a, b, Banach 1980) that fields on $T\otimes M$ are pulled back to fields on $T\otimes\tilde{M}$ which satisfy

$$\phi(\gamma x)=a(\gamma)\phi(x) \qquad \forall x\in T\otimes\tilde{M}, \forall\gamma\in\Gamma \tag{1}$$

where Γ is the discrete group of isometries of $\tilde{M}$ which yields M by point identification and $a(\Gamma)$ is a representation of Γ in a compact matrix group (with indices suppressed). We denote such fields on $T\otimes\tilde{M}$ by $\phi=\phi^a$ and recall also that (for $|\Gamma|$ finite) there is an orthogonal split in the space of fields so that for any ϕ we have uniquely

$$\phi=\phi^a+\phi^\perp \tag{2}$$

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where ϕ^a and $\phi^\perp$ are the automorphic and perpendicular parts of ϕ respectively. We finally remember that the propagators will also respect the above split:

$$\Delta = \Delta^a + \Delta^\perp \tag{3}$$

$$\Delta f^a = \Delta^a f = \Delta^a f^a \qquad \Delta f^\perp = \Delta^\perp f = \Delta^\perp f^\perp \qquad \Delta^a f^\perp = \Delta^\perp f^a = 0 \tag{4}$$

where index summations and space–time integrations have been suppressed in (4).

We now consider the vacuum generating functional $Z^a(J)$ on $T \otimes \tilde{M}$ given by inserting an automorphic δ functional $\delta(\phi - \phi^a)$ into the vacuum generating functional $Z(J)$ on $T \otimes \tilde{M}$ to restrict the integration to automorphic fields—identifying the result with the vacuum generating functional on $T \otimes M$:

$$\begin{aligned} Z^a(J) &= \int [\mathrm{d}\phi][\mathrm{d}\lambda] \exp\{\mathrm{i}[S(\phi) + \lambda(\phi - \phi^a) + J\phi]\} \\ &= \int [\mathrm{d}\phi][\mathrm{d}\lambda] \exp\{\mathrm{i}[S(\phi) + (\lambda - \lambda^a)\phi + J\phi]\} \\ &= \int [\mathrm{d}\lambda] \exp\left[(\lambda - \lambda^a)\frac{\delta}{\delta J}\right] \int [\mathrm{d}\phi] \exp[\mathrm{i}(S(\phi) + J\phi)] \\ &= \Pi^a\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right) Z(J). \end{aligned} \tag{5}$$

We thus notice that we can get $Z^a(J)$ by applying an operator Π^a to $Z(J)$. Further, we can write (note the automorphic analogue of the Fadeev–Popov trick)

$$\begin{aligned} \Pi^a\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right) &= \int [\mathrm{d}\lambda] \exp(\lambda - \lambda^a)\frac{\delta}{\delta J} = \int [\mathrm{d}\lambda^a] \int [\mathrm{d}\lambda^\perp] \exp \lambda^\perp \frac{\delta}{\delta J} \\ &= N \int [\mathrm{d}\lambda^\perp] \exp \lambda^\perp \frac{\delta}{\delta J}. \end{aligned} \tag{6}$$

With the divergent integral over automorphic fields removed, Π^a becomes a well behaved operator; indeed, we find after translating the integration variable and changing its name that

$$Z^a(J) = \int [\mathrm{d}J^\perp] Z(J^a, J^\perp) \tag{7}$$

where the automorphic and perpendicular degrees of freedom of $Z(J)$ are written explicitly separated in (7). Noting that in perturbation theory $Z(J)$ is rendered integrable (in the Euclideanised sense) by the factor $\exp(\frac{1}{2}\mathrm{i}J\Delta J)$, application of Π^a is a finite operation and thus the proof of renormalisability is reduced to the following trivial remark. Suppose $Z(J)$ has been made finite in the limit $\epsilon \to \epsilon_0$ (where ϵ is some regularisation parameter) by using divergent, bare, ϵ-dependent quantities in the action functional, then $Z^a(J)$ will be finite in the same limit since Π^a is independent of ϵ.

Some further remarks are in order.

Since Π^a is independent of ϵ, the counterterms needed in $Z^a(J)$ will be exactly those already present in $Z(J)$—the renormalisation prescription is left unaffected by passing to $T \otimes M$. Now we can write

$$Z(J) = \Sigma_I \exp(\tfrac{1}{2}\mathrm{i}J\Delta J)$$

where

$$\Sigma_I = \exp\left(\mathrm{i}S_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right)$$

and S_I is the non-quadratic part of the action functional. It is clear that Π^a and Σ_I commute so that

$$Z^a(J) = \Sigma_I \exp(\tfrac{1}{2}\mathrm{i}J^a\Delta^a J^a). \tag{8}$$

This tells us that to calculate a process on $T \otimes M$ we need to calculate the same set of diagrams using the same set of vertices, but using only the automorphic parts of the covering space propagators. Now Δ^a can be written as an image sum of covering space propagators (Banach and Dowker 1979a, b, Banach 1980)

$$\Delta^a(x, x') = \sum_{\gamma \in \Gamma} \Delta(x, \gamma x')a(\gamma), \tag{9}$$

which in the troublesome coincidence limit splits into the sum of the divergent coincidence limit $\Delta(x, x)$ plus the finite parts $\Delta(x, \gamma x)a(\gamma)$. In multi-loop processes this state of affairs gives rise to nonlocal infinities of the form $\Delta(x, x)\Delta(x, \gamma x)a(\gamma)$, etc, which are not cancelled by counterterms. Our analysis, however, guarantees that when all relevant diagrams are summed, all such nonlocal infinities automatically cancel without additional effort. Similar facts have already been noticed in certain specific calculations (Ford 1980, Birrell and Ford 1980, Toms 1980a, b).

If one regards automorphic field theories not as theories in themselves but as representatives of individual sectors of Isham's twisted field theories (Isham 1978a, b, Avis and Isham 1978b), then one has to make the vacuum generating functional invariant under large as well as small gauge transformations. This can be done, for example, by taking linear combinations or products of sectors Z^a over a (finite maximal) set of inequivalent a (Avis and Isham 1979, Chockalingham and Isham 1980). Clearly the renormalisability argument goes through with little change in these cases.

We note that Π^a does not commute with the taking of logarithms. This means we cannot repeat the renormalisability proof for the connected or 1PI generating functionals. It is easy to see why. From (8) we see that the effect of Π^a in the diagrammatic expansion of $Z(J)$ is to replace each propagator by its automorphic part; thus Π^a, being an integral over $J^\perp$, can only do this *directly* if the relevant propagator has a source attached. If not, the relevant cancellation is produced by an associated diagram wherein the propagator in question is broken and the two additional free ends are saturated with sources. Hence the need for disconnected diagrams.

Since Π^a commutes with functional derivatives (any 'boundary contributions' to $\int[\mathrm{d}J^\perp]$ being zero by the damping effect of the $\exp(\frac{1}{2}\mathrm{i}J\Delta J)$ factor) we can equally well apply our analysis to the (disconnected) Green functions, $\delta^n Z/\delta(\mathrm{i}J)^n$. We illustrate this with a simple example, namely the two-point function of a free scalar field in an arbitrary simply connected background geometry. With arbitrary J it is (variables suppressed)

$$G_2(J) = [-\mathrm{i}\Delta + (\Delta J)^2] \exp(\tfrac{1}{2}\mathrm{i}J\Delta J). \tag{10}$$

We require essentially

$$\exp(\tfrac{1}{2}\mathrm{i}\,J^a\Delta^a J^a)\int [\mathrm{d}J^\perp](\Delta^a J^a+\Delta^\perp J^\perp)^2\exp(\tfrac{1}{2}\mathrm{i}J^\perp\Delta^\perp J^\perp)$$

$$=\exp(\tfrac{1}{2}\mathrm{i}J^a\Delta^a J^a)\int [\mathrm{d}J^\perp]\Big(\Delta^a J^a+\frac{\delta}{\delta(\mathrm{i}\eta^\perp)}\Big)^2 \times\exp(\tfrac{1}{2}\mathrm{i}J^\perp\Delta^\perp J^\perp+\mathrm{i}\eta^\perp\Delta^\perp J^\perp)|_{\eta=0}. \tag{11}$$

This can be evaluated trivially and when we insert it into (10) we find as expected

$$\Pi^a G_2(J)=G_2^a(J)=[-\mathrm{i}\Delta^a+(\Delta^a J^a)^2]\exp(\tfrac{1}{2}\mathrm{i}J^a\Delta^a J^a). \tag{12}$$

Nothing changes much when we consider interacting theories, except that we have to keep track of the space–time variables inside the vertices. However, the magnitude of the expressions generated rapidly becomes gigantic. For instance, the calculation of the two-point function in ϕ^4 theory to first order involves ten diagrams and produces expressions of several hundred terms. For this reason the calculation is best done on a computer, and the author has done some sample calculations using the algebraic manipulation language REDUCE, which proved ideal for the purpose. The cases considered were, in ϕ^4 theory, Z to first and second order, $\delta^2 Z/\delta(\mathrm{i}J)^2$ to first order; in ϕ^2 theory (i.e. a quadratic theory with a quadratic perturbation) $\delta^2 Z/\delta(\mathrm{i}J)^2$ to second order. As with all algebraic manipulation programs, large amounts of core are required and the largest of the calculations mentioned reaches the practical limits of even a big machine so that it is not really possible to go beyond second order. In all cases Π^a behaved as expected, replacing all propagators by their automorphic parts.

To conclude, we have seen that (at least in the $|\Gamma|$ finite case) renormalisability for a multiply connected space is an essentially trivial issue—provided, of course, that it is under adequate control for the covering space.

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